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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Militant Mathematics

We sometimes wonder if national leaders in the cause would not be using more effective strategy toward the rebuilding of mathematics in American secondary schools if they should speak and write less about the need of militancy in mathematical activities and more about the need for constructive or experimental planning where mathematics is concerned. It is a questionable psychology that is constantly publishing to the world, embracing, as the world does, multitudes of the uninformed and the indifferent, that mathematical studies are being run out of the schools, or that its very foundations are crumbling.

Sometimes the weakest cause is the most militant. Militant attitudes easily beget hostile ones. On the other hand, mathematics, known by the properly informed to possess the highest tool, culture and discipline potentials, needing not to be defended, should attain its significant advances, not by militant aggressiveness, but by deliberate programs aiming to *inform* and to *demonstrate*. It is better for the mathematical missionary who would convert a school supervisor to the cause of algebra or geometry to flourish no stick or issue no challenge. To the uninformed administrator this will be but invitation to attack. Rather should the attitude of a true mathematical evangel be summed in the language, "Come and see for yourself what mathematics can do; permit us to SHOW you."

We do need to say that the fruitage of such an attitude and hence the making of a convert must depend upon the mathematical demonstrator's ability to make good. And here is disclosed to view our greatest weakness. Who can estimate the vast number of those in the present generation of mathematics teachers that are unable to make good—yes, and who themselves are without faith? How many of us can confidently assume the task of turning out within a proper time-period the student whose orderly, logical and accurate thinking is largely to be credited to our own mathematical programs?

Professor W. W. Hart, of the University of Wisconsin, sounded a keynote to real mathematical advance when, in a paper read before the recent meetings of M. A. of A. and N. C. T. M. in St. Louis he declared that henceforth he stands for a six-year mathematical curriculum under the college level. No conscious transfer of the fundamental disciplines of mathematical study can be satisfactorily made in brief and broken periods of time. We have the faith to believe that in an unbroken six-year pre-collegiate school period of demonstration we could convert the world, if only our teachers have vision, inspiration, knowledge.

S. T. SANDERS.

On Certain Systems of Conics Satisfying Four Conditions

By H. E. FETTIS
Dayton, Ohio

1. *Remarks.*

The study of systems of conics which satisfy four conditions has long been a favorite in pure geometry. Among the more notable devices which have been employed to investigate the properties of such systems, we may mention the theory of involution, Chasles' method of characteristics, and central projection. It may also be approached algebraically, but in this article we will rely almost exclusively on projective methods.

The reader of this article should be fairly well versed in the concepts of projective geometry. We shall, as is the custom in pure geometry, deal equally with real and imaginary, finite and infinite elements, and when mention is made of points of lines, it is to be understood that they may be any of these.

2. *Prerequite Theorems.*

Theorems on imaginary elements:

(a) A conic is cut by an arbitrary line in its plane in two points. These may be real, coincident, or imaginary.

(b) From a point in the plane of a conic, two tangents may be drawn. These may be real, coincident, or imaginary.

(c) The join of two imaginary points may be real.

(d) The intersection of two imaginary lines may be real.

(e) The join of a real and an imaginary point is imaginary.

(f) The intersection of a real and an imaginary line is imaginary.

(g) Through an imaginary point there passes one and only one real line.

(h) On an imaginary line there lies one and only one real point.

(i) All similar and similarly placed conics cut the line at infinity in the same two points. These are real, coincident, or imaginary according as the conic is an hyperbola, a parabola or an ellipse.

(j) In particular, all circles cut the line at infinity in the same two imaginary points, the "imaginary circular points at infinity."

(k) If F is the focus of a conic, A and B the imaginary circular points at infinity, then FA and FB are tangents to the conic.

Theorems on Projection:

(a′) Any line (and therefore any two points) may be projected to infinity.

(b′) Any conic may be projected into a circle.

(c′) Given a conic, and a line in its plane, we may project the line to infinity, and at the same time project the conic into a circle. (Poncelot's theorem).

(d′) Any two conics may be projected into circles, by projecting one into a circle, and one of their chords of intersection to infinity.

(e′) A system of conics through four points may be projected into a system of coaxial circles, by projecting two, and hence all into circles.

3. *It is from this last result that we shall deduce the properties of a system of conics through four points. We see that such a system will contain many ellipses, many hyperbolae, (one rectangular) two parabolas, and three pairs of intersecting lines.*

The three pairs of lines are called the *degenerate conics* of the system, and they form a complete quadrangle (the *radical quadrangle*) *any side of which may be thought of as corresponding to the radical axis of the system of circles from which we are deriving the properties.*

The deriving of the sought properties now reduces itself to a mechanical operation. We write on the left a theorem (usually a well known one) concerning coaxial circles, and from this we obtain, by projection, another concerning a system of conics through four points, which is placed in the parallel column on the right:

(a) The polars of a fixed point with respect to the members of a coaxial system pass through a fixed point.	(a′) The polars of a fixed point with respect to the members of a system of conics through four points pass through a fixed point.
(b) A common tangent to two circles of a coaxial system is cut harmonically by any third.	(b′) A common tangent to two members of a system of conics through four points is cut harmonically by any third.

(c) There are at most two circles of a coaxial system which touch an arbitrary line.

(c′) There are at most two members of a system of conics through four points which touch an arbitrary line. (Hence only two parabolas).

(d) If from any point on the radical axis of a coaxial system tangents be drawn to the members their points of contact lie on a circle.

(d′) If from any point on the radical quadrangle of a system of conics through four points, tangents be drawn to the members, their points of contact lie on a conic.

(e) The circles thus determined are themselves coaxial.

(e′) The conics thus determined pass, themselves, through four points.

(f) The two systems of circles involved in this manner are "conjugate". That is, each may be formed by drawing tangents to the other from the latter's radical axis.

(f′) The two systems of conics involved in this manner are "conjugate". That is, each may be formed by drawing tangents to the other from the latter's radical quadrangle.

4. *The theorems which we obtain in this manner are of a most general nature, and may therefore, be specialized to yield still other important results. For example, we may form the reciprocal of any one of them. Thus the reciprocal of 3(a′) is that*

The poles of a fixed line with respect to the members of a system of conics touching four lines lie on a line.

In particular,

The centers of all such conics lie on a line.

This last result is itself of some consequence. Of all conics which touch four lines, many are ellipses, many are hyperbolas, one is a parabola, and three are pairs of points. Hence the following well known theorem is included:

The midpoints of the diagonals of a complete quadrilateral lie on a line.

Again, we have said that coaxial circles pass through the same two points at infinity. Hence the line at infinity may be regarded as one side of the radical quadrangle of a coaxial system, and 3(d′) may be thrown into the following form:

If a system of parallel lines touch a system of coaxial circles, their points of contact lie on a conic.

Or, what is equivalent:

If a system of parallel diameters be drawn in a system of coaxial circles, their end points will lie on a conic. (See figure below.)

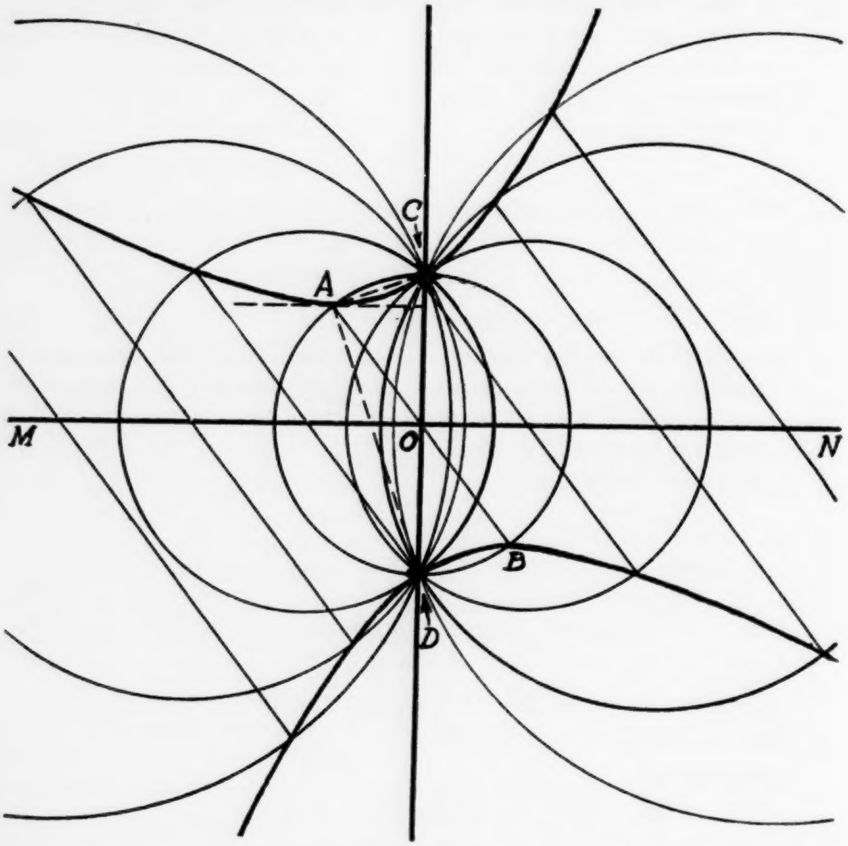


FIGURE 1.

5. The conics defined by the last theorem have many interesting properties which we will briefly summarize.

We first observe that the conics will have the same relation to the given coaxial system as to the conjugate system. For if we draw a system of parallel diameters in one, their end points obviously will be the points of contact of a system of parallel tangents to the members of the other. We may therefore confine our attention to whichever

type gives the simplest treatment; it is found that it is most convenient to use a system with real, rather than imaginary intersections.

Secondly, we see that the conics will pass through the limiting points, when these exist, but otherwise, through the points common to all the members of the system.

Let MN be the line of centers of the circles, C and D the common points, O the intersection of MN and CD, AB that one of the parallel diameters which passes through O.

AB is a diameter of the conic, and MN bisects all chords parallel to AB. Hence AB and MN are conjugate diameters.

All the conics thus generated are rectangular hyperbolae. This may be established in several ways. For example, by making use of the well known theorem that if a conic passes through the vertices and orthocenter of a triangle, it is a rectangular hyperbola. In the case of a right triangle, this theorem reads as follows: If a conic passes through the vertices of a right triangle, and touches the altitude at the right angle, it is a rectangular hyperbola. Consider now the right triangle DAC. The tangent at A is parallel to MN, the diameter conjugate to AB. Hence the tangent at A is the altitude at A, thus making the conic a rectangular hyperbola, as was to be shown.

6. *Conditions for real existence of the degenerate conics and common self-polar triangle.*

A system of conics through four real points obviously has three real degenerate conics. In the case of two real and two imaginary points, there is only one real degenerate conic; this consists of the join of the two real points, and the join of the two imaginary points.

In the case of four imaginary points, there is only one real degenerate conic. For were there more, we would have more than one real line through an imaginary point, which contradicts 2(g).

The diagonal triangle of the radical quadrangle is called the common self polar triangle of the system, because it is self polar with respect to any member. And in addition, it is also the diagonal triangle for the quadrilateral formed by the common tangents. Hence for two conics having either four real points in common, or four real tangents, the common self polar has real existence, and can be constructed. For two real points, one vertex and the opposite side of the common self polar triangle can be constructed.

To construct the real degenerate conic of a system having imaginary intersections, draw a pair of common tangents (either transverse or direct) intersecting at P. If a line through P cuts the conics in A, B, C, and D, then the tangents A and D, B and C, A and C, B and

D intersect at four points which, joined, give the desired degenerate conic. For two circles, this construction becomes the familiar construction for the radical axis, from which the above may be derived by projection.

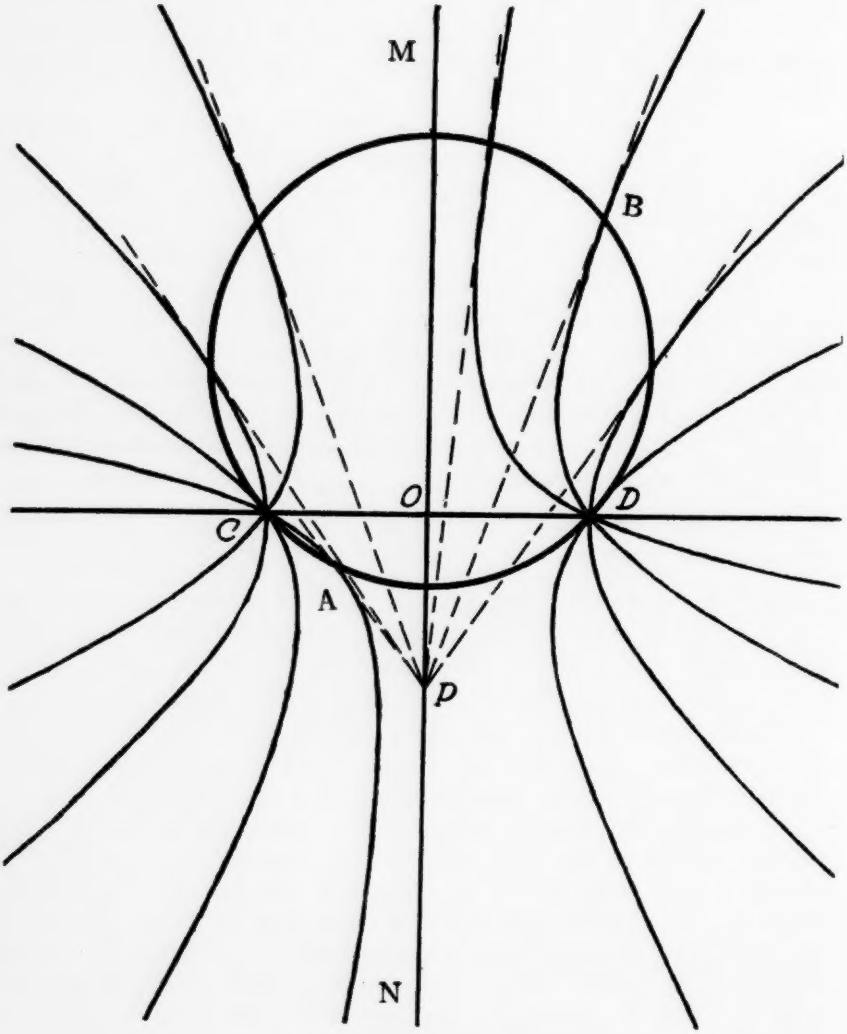


FIGURE 2.

By drawing the parallel diameters in different directions, we obtain different hyperbolae, which, by 3(e′) constitute a system of conics through four points. The members have already two real points,

namely C and D, in common, and must therefore have two imaginary points in common. This is enunciated in the following theorem that

If a number of rectangular hyperbolae be concentric, and pass through the same two real points, they pass also through the same two imaginary points.

It is a result illustrative of the type which could not be readily conceived or approached otherwise, except perhaps by analytic means.

The axes of any member can at once be constructed by bisecting the angles AOC and AOD in Figure 2. The asymototes are the bisectors of the angles between the conjugate diameters AB and MN.

The figure 3, on the succeeding page, which shows several members of this family of hyperbolae, enables us to obtain some sort of a conception of the form of a system of conics with imaginary intersections. In this figure are illustrated also some of the properties of this family which can be deduced at once as special cases of the theorems in article 3. The lines MN and CD of Figure 2 constitute the only real degenerate conic of this system.

Note: We may generalize this theorem by substituting for the words coaxial circles the words similar and similarly placed conics through two points. The same general argument will hold, but the conics will not in general be rectangular hyperbolae.

7. Special Cases.

We conclude by noting some various other systems of conics which may be found as particular cases of those already studied.

According to 2(c) if several conics have one focus in common, they have two imaginary tangents through this focus in common, and if they have two foci in common, they have four common imaginary tangents. Therefore a system of confocal concis is nothing more than a special case of a system of conics touching four lines, and the reciprocals of all the theorems given in the right hand column of article 3 will apply to confocal conics. We believe it unnecessary to go into detail with these properties, as they can present no real difficulty to the reader who wishes to investigate them.

Let us now consider the case in which some of the points have coincided. Let the four given points be A, B, C, and D, and let A and B coincide at A′, C and D at C′. The result will be a system of conics having double contact at A′ and C′, and the properties of such a system may be found by specializing the results of article 3. But such a system has still further properties due to the fact that the members may be thought of, not only as having two pairs of coincident points in common, but also two pairs of coincident tangents

at these points in common. Thus not only do all the properties of Article 3 apply, but also their reciprocals. Here again we leave the reader to work out the details which in this case are even more abundant, because of the "hybrid" nature of the system. It might be

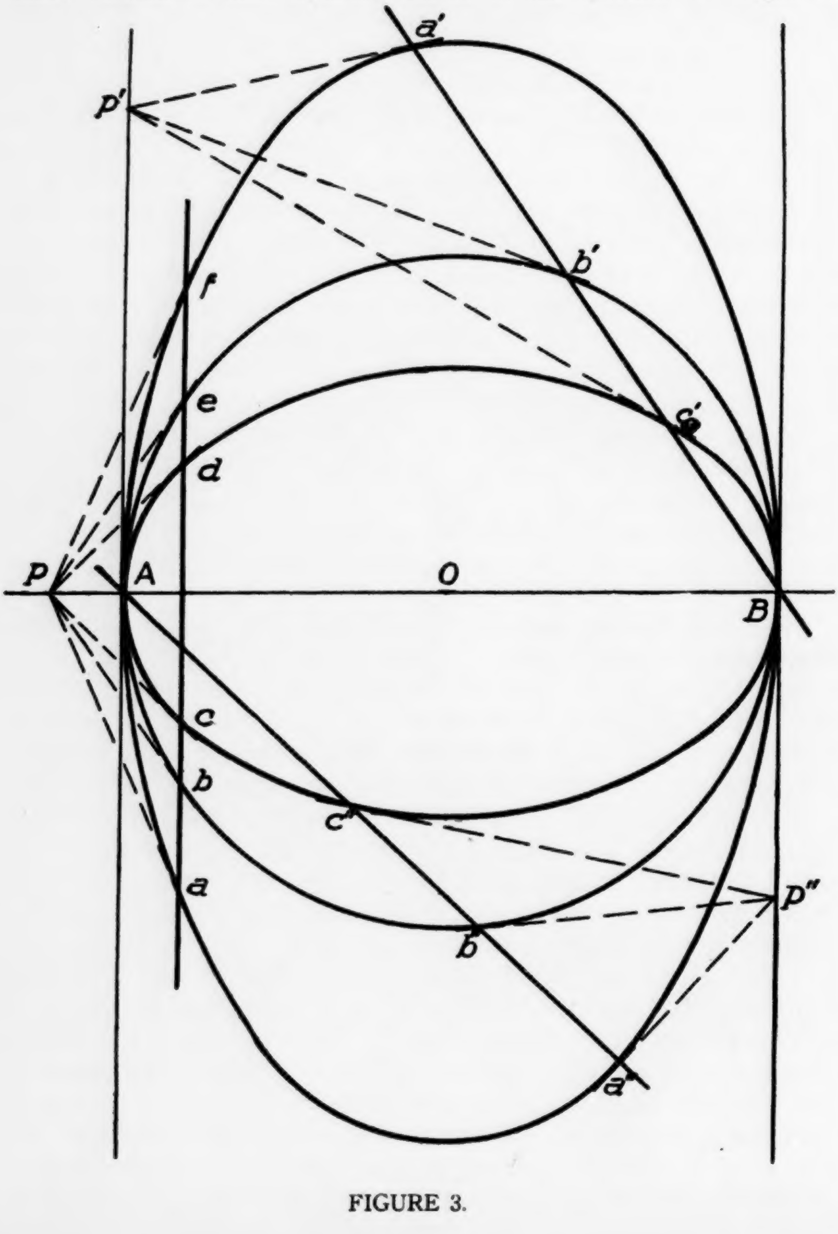


FIGURE 3.

remarked, however, that the conics determined by 3(d′) reduce to straight lines.

Again, we have said that similar and similarly placed conics pass through the same two points, because their asymptotes are parallel. If, now, these conics should, in addition, be concentric, the asymptotes will be coincident, and we learn that similar, similarly placed, and concentric conics are to be regarded as having double contact at infinity, and will therefore obey all the properties of a system of conics having double contact.

As before, we encounter some difficulty in constructing geometrically a system of conics having double contact. The construction is theoretically possible, but not convenient in practice. If we draw several conics with a common axis, AB (as on the succeeding page) we obtain a system having double contact at A and B, and this special case may be used to illustrate the properties of the more general case. In any case, the tangents at A and B and the chord of contact, AB constitute the degenerate conics of the system. By taking three points P, P′ and P″, one on each of the three degenerate conics of the system, and drawing tangents from them to the members, we obtain the locus of the points of contact as three straight lines. These pass respectively through A, B, and the intersection of the common tangents at A and B, viz, through the poles of each of the three degenerate conics. The reciprocal of 3(d′) may be illustrated on this figure by considering A and B as degenerate point conics. In this case, however, we must substitute for "envelope a conic" the words "pass through a fixed point."

Still another type of system which may be used as a practical illustration of these properties may be constructed by taking any point, P, of a given conic, drawing through it chords in various directions, and dividing these into the same number of equal parts. These points of division give a system of conics, similar and similarly placed, which touch at P, and which therefore constitute a system through four points.

8. *Concluding remarks.*

We believe it superfluous to go into detail concerning other systems of conics which are more easily treated by means of the method of characteristics; lack of space itself forbids a more detailed investigation of these questions. It has been the purpose of the writer, however, to present to those familiar with the subject what are perhaps some new developments, and also to prompt the study of this branch of mathematics by those who have not already done so. The latter

class will observe that all the results were obtained by pure reasoning, rather than by observation, and that drawings were employed to illustrate, and not to deduce these results. It is this feature of pure geometry which exalts it above the elementary methods of Euclid. By it are avoided what are termed by J. W. Young in his "*Teaching of Mathematics*," "mouse trap proofs," which, though they may prove with all rigor the theorem under consideration, give the reader little or no idea of how the discoverer of the theorem may have first arrived at the principle result.

COMMUNICATION TO THE EDITOR

Sir: Some question has arisen in connection with Rule III, p. 21, Vol. 10, 1935, of the MAGAZINE, in my paper on Significant Figures. This rule deals with addition and subtraction, and with multiplication and division. It is, of course, as the illustration in addition employed shows, the number of digits to the right of the decimal in the least precise number appearing in an addition or subtraction with which we are concerned; this should be distinguished from the case of multiplication or division where the number of significant figures in the least precise factor governs the precision of the product or the quotient. This point regarding addition or subtraction obviously must also be remembered in adding or subtracting logarithms where we retain as many digits to the right of the decimal in the mantissa as there are significant figures in the factor.

CARROLL W. GRIFFIN.

Application of Elliptic Functions to Certain Problems in Plane Cubics

By MARIAN E. DANIELLS
Iowa State College

Several well-known theorems concerning plane cubics may be obtained readily by the use of the Weierstrass "p" function. This doubly periodic function is defined by the equation

$$(1)\qquad p(u) = \frac{1}{u^2} + \sum{}' \left(\frac{1}{(u-w)^2} - \frac{1}{w^2} \right) \qquad \begin{array}{l} w = 2m\omega + 2n\omega' \\ m = n = o \text{ excluded} \end{array}$$

and since $\sum{}'$ is holomorphic in the neighborhood of the origin it can be expanded in a power series good inside a circle with radius equal to the lesser of the two quantities $|2\omega|$ and $|2\omega'|$

$$(2)\qquad p(u) = \frac{1}{u^2} + \frac{g_2}{20} u^2 + \frac{g_3}{28} u^4 + \ldots\ldots\ldots$$

where g_2 and g_3 are the invariants of the p function and have the values

$$g_2 = 60 \sum{}' \frac{1}{w^4} \text{ and } g_3 = 140 \sum{}' \frac{1}{w^6}.$$

$$(3)\qquad p'(u) = -\frac{2}{u^3} - 2\sum{}' \frac{1}{(u-w)^3}$$

This function also is doubly periodic, has roots ω, ω' and $\omega+\omega'$ and can be expanded in a series of the form

$$(4)\qquad p'(u) = -\frac{2}{u^3} + \frac{g_2}{10} u + \frac{g_3}{7} u^3 + \ldots\ldots\ldots$$

In the theory of elliptic functions it is shown

(1) that to every system of complex numbers ω and ω' whose ratio is real corresponds a completely determined elliptic function, p (u), with periods 2ω and $2\omega'$

(2) that p (u) satisfies the differential equation
$[p'(u)]^2 = 4p^3(u) - g_2\, p(u) - g_3$

(3) that to every system of values of the invariants g_2 and g_3 subject to the condition $g_2^3-27g_3^2\neq o$ corresponds an elliptic function.

In analytic geometry it is shown that every third order curve can be reduced to the form

(6) $$y^2=4x^3-g_2x-g_3$$

where g_2 and g_3 are any given real constants; and when $g_2^3-27g_3^2\neq 0$, the cubic has no double points.

The following theorem is basic:

It is always possible to express the coordinates of a point on curve (6) as a function of a single parameter *u* by setting

(7) $$x=p(u) \qquad y=p'(u)$$

Proof. To each value of *u* there corresponds a point (real or imaginary) of the curve, since p and p′ are one-valued functions; this point remains the same when multiples of the periods 2ω and $2\omega'$ are added to *u*.

Conversely, to each point (x, y) of the curve corresponds, in an elementary parallelogram, *only one* value of u. When only x is given, the equation $x=p(u)$ gives two values of u, u_1 and $-u_1$, and all homologous values. Since $p'(u)$ is odd, to these two systems of values for u correspond two values of y equal and of opposite sign. These are the two values that are obtained from (6). Therefore theorem is proved.

The first theorem concerning cubics which is pertinent to this discussion is as follows:

A necessary and sufficient condition that three points of a cubic shall be collinear is that $u_1+u_2+u_3=2n\omega+2n'\omega'$ where n and n′ are integers and u_1, u_2 and u_3 the parameters of the given points.

Proof. The values u_1, u_2, u_3 situated in an elementary parallelogram are the roots of

$p'(u)-a\,p(u)-b=o$, $y-ax-b=o$ being the equation of the line.

The first member of this equation is an elliptic function of order 3 and has in an elementary parallelogram three zeros u_1, u_2 and u_3 and a pole of order 3 for $u=o$. Therefore by Liouville's Theorem

(8) $u_1+u_2+u_3+o=2n\omega+2n'\omega'$, n and n′ integers.

To prove the condition sufficient:

Let M_1, M_2, M_3 be the points corresponding to u_1, u_2, u_3. Join M_1 and M_2 by a straight line and call the point where this cuts the cubic M_3' with parameter u_3'.

Then $u_1+u_2+u_3'=2m\omega+2m'\omega'$.

Comparing this with (8) it is evident that u_3' differs from u_3 by a multiple of a period. Therefore M_3' coincides with M_3 and the condition is sufficient.

1. Application to finding the number of tangents thru a given point on a cubic. Through a point on the cubic it is possible, in general, to draw four tangents distinct from the tangent at the point under consideration.

Proof. Consider the tangent to the cubic at the point for which the parameter is u. This tangent cuts the curve again: Let v be the parameter of this point: then

$$v+2u=2n\omega+2n'\omega'$$

$$u=-\frac{v}{2}+\frac{2n\omega+2n'\omega'}{2}$$

In this formula, one can give to n and n' all integral values: but two values of u that differ by any multiple of 2ω and $2\omega'$ give the same point on the curve. So it is sufficient to give n and n' the values o and 1 associated in all possible ways. Thus there are four values of u

$$-\frac{v}{2},\quad -\frac{v}{2}+\omega,\quad -\frac{v}{2}+\omega'\quad -\frac{v}{2}+\omega+\omega'$$

2. As another application, let us find the points of inflexion on a cubic.

If u is the parameter of a point of inflexion, since the inflexion tangent cuts the curve in three coincident points $u_1=u_2=u_3=u$

$$u=\frac{2n\omega+2n'\omega'}{3}$$

where n and n' have all integral values. But since two values of u that differ by a period are the same it is sufficient to give n and n'

the values o, 1, and 2, associated in every possible way. Thus we find nine points of inflexion, the parameters of which are

$$u_{00}=0 \qquad u_{01}=\frac{2\omega'}{3} \qquad u_{02}=\frac{4\omega'}{3}$$

$$u_{10}=\frac{2\omega}{3} \qquad u_{11}=\frac{2\omega+2\omega'}{3} \qquad u_{12}=\frac{2\omega+4\omega'}{3}$$

$$u_{20}=\frac{4\omega}{3} \qquad u_{21}=\frac{4\omega+2\omega'}{3} \qquad u_{22}=\frac{4\omega+4\omega'}{3}$$

Only three of these points are real.

The straight line thru two points of inflexion

$$\frac{2m_1\omega+2m_2\omega'}{3} \quad \text{and} \quad \frac{2m_1'\omega+2m_2'\omega'}{3}$$

passes thru a third point

$$-\frac{2(m_1+m_1')\omega+2(m_2+m_2^1)\omega'}{3}$$ which is also a point of inflexion.

The number of straight lines which meet the cubic in three points of inflexion is

$$\frac{9.8}{3.2}=12$$

The second theorem concerning cubics is:

The condition that 3n points of a cubic shall be on a curve of order n is

$$u_1+u_2+u_3 \ldots\ldots\ldots\ldots +u_{3n}=2n\omega+2n'\omega'$$

First consider the condition that six points of a cubic shall be on a conic.

If the cubic $x=p(u)$, $y=p'(u)$ cuts the conic

$$Ax^2+2\ By+Cy^2+2\ Dx+2\ Ey+F=0,$$

the parameters of the points of intersections are the roots of

$$A[p(u)]^2+2\,Bp(u)p'(u)+C[p'(u)]^2+2\,Dp(u)+2\,Ep'(u)+F=0.$$

The first member of this equation is a doubly periodic function, which in an elementary parallelogram that includes the origin, has *zero* as a pole of order 6 and no other pole. So the equation has 6 roots: and by Liouville's Theorem

$$u_1+u_2+u_3+u_4+u_5+u_6=2n\omega+2n'\omega'$$

To show that the condition is sufficient. If a conic passes thru 5 points on a cubic $u_1+u_2+u_3+u_4+u_5+u_6=2n\omega+2n'\omega'$. Therefore u_6' must be congruent to u_6. Similarly the condition for 3n points of a cubic being on a curve of order n is $u_1+u_2.+u_3\ldots\ldots+u_{3n}\equiv 0$. The symbol $\equiv$ indicates the equality holds with multiple of a period. For example, another cubic cuts the given cubic in *nine* points which must be subject to *one* condition since through nine given points, there passes, in general, only one cubic. The preceding theorem expresses this condition in the simplest way.

This theorem has many geometric applications of which we shall give a few examples.

APPLICATIONS

1. When six of the nine points of intersection of two cubics lie on a single conic, the other three points of intersection are on a straight line. If u_1, u_2, $u_3,\ldots\ldots u_9$ are the parameters of the points of intersection of the cubics, then $u_1+u_2+u_3+\ldots\ldots+u_9\equiv 0$ by above theorem. If the first six of these points lie on a conic then

$$u_1+u_2\ldots\ldots+u_6\equiv 0$$

From these two relations it follows that

$$u_7+u_8+u_9\equiv 0$$

which is the condition for these points being in a straight line.

2. If we consider a variable conic passing through four fixed points on a cubic, the straight line that joins the two movable points of intersection passes through a fixed point of the cubic.

Let u_1, u_2, u_3, u_4, u_5, u_6 be the parameters of the six points of intersection, the first four being fixed.

Then $u_1+u_2+u_3+u_4=v$ (v a constant).

If these six points of the cubic are on a conic

$$v+u_5=u_6\equiv 0$$

This shows that the points whose parameters are u_5, u_6, and v are on a straight line.

Applications to Curves of Contact.

Curves of contact are curves which have several coincident points of intersection with the cubic.

3. Let us consider conics that are three times tangent to the cubic. If the parameters of the points of contact are u_1, u_2, u_3 then

$$2u_1+2u_2+2u_3=2n\omega+2n'\omega'$$

$$u_1+u_2+u_3=n\omega+n'\omega'$$

It is sufficient to give n and n' the values o and 1.

If $n=n'=o$ the relation becomes

$$u_1+u_2+u_3=o$$

which expresses the condition for the three points being on a line. This is the case when the conic reduces to two lines: let us discard this case.

There remain three families of conics corresponding to the relations

$$u_1+u_2+u_3\equiv\omega \qquad \text{1st family}$$

$$u_1+u_2+u_3\equiv\omega' \qquad \text{2nd family}$$

$$u_1+u_2+u_3\equiv\omega+\omega' \qquad \text{3rd family}$$

One can then choose arbitrarily two of the points of contact for each conic of a family. Let us take a conic of the first family for example: if a conic passes through the three points of contact u_1, u_2, u_3, it meets the cubic in three other points u_1', u_2', u_3' and we have

$$u_1+u_2+u_3+u_1'+u_2',\ +u_3'=2n\omega$$

Therefore $\quad u_1'+u_2'+u_3'\equiv(2n-1)\omega$

and we see that the three new points are also the points of contact of a conic, belonging to the same family, and three times tangent to a cubic. Similar results follow for conics of the second and third families.

4. Again let us investigate those points of the cubic where the osculatory conic has contact of the fifth order, or, what amounts to the same thing, the conics which cut the cubic in six coincident points.

For the parameter of a point of contact

$$6u = 2n\omega + 2n'\omega'$$

$$u = (n/6)2\omega + (n'/6)2\omega'$$

Each of the integers n and n' can take all of the values from 0 to 5 which gives $6^2 = 36$ points. Among these points are the nine points of inflexion which were obtained by considering the inflexion tangents as two lines so

$$6^2 - 3^2 = 27$$

is the number of points of contact of genuine osculatory conics.

These points, called sextactiques, are six by six on the conic.

Mechanically Described Curves

By ROBERT C. YATES
University of Maryland

I

The fundamental problems in the study of Analytic Geometry are those of determining the curve traced out by a point moving in accordance with some law, and its converse. Once the law is stated in symbolic form it usually becomes lost to the student and his interest unfortunately is consumed in the algebraic reduction. This is to be deplored since the laws give rise to the characteristic properties and are, in fact definitions of the curves themselves.

It is suggested then that the teacher of Analytics, whenever possible, make use of some means to emphasize the mechanical description of curves. The interest of the student—especially that of the engineer—will show a sharp increase, most particularly during the time devoted to polar coordinates.

II

To classify all curves and give mechanical systems that would describe them is an impossibility. However, much can be done in the field of Freshman mathematics and the following are given only with the idea of forming a nucleus of suggestions.

The linkage* that will describe a straight line was invented by Peaucellier in 1864, and shown in Fig. 1.

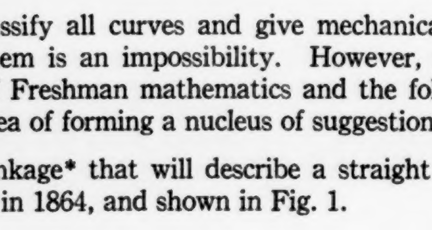


Fig. 1

*An entertaining account of the interest manifested in pure link motion in the latter part of the last century may be found in the 1919 issues of the *Scientific Monhtly.*

The points O and A are fixed to the plane. OC = OB and PBQC is a rhombus. We have:

$$(OB)^2 = (OR)^2 + (BR)^2$$
$$(PB)^2 = (PR)^2 + (BR)^2$$

from which:

$$\begin{aligned}(OB)^2 - (PB)^2 &= (OR)^2 - (PR)^2\\ &= (OR+PR)(OR-PR)\\ &= (OP)(OQ).\end{aligned}$$

Thus the product (OP)(OQ) is at all times constant, say k^2. If the distance OA be taken equal to AP then the point Q will proceed along a straight line as P travels on its circle. Without the bar AP the instrument may be used to find curves inverse with respect to a circle.

For the conics, the trammel and string methods may be found in almost every text and are omitted here because of their familiarity. An interesting system for drawing the ellipse is shown in Fig. 2.

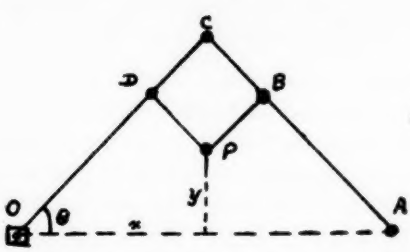


Fig. 2.

Here OC = AC = b, and PDCB is a rhombus with side a. From the figure:

$$x = (b-a)\cos\theta + a\cos\theta = b\cos\theta$$

$$y = (b-a)\sin\theta - a\sin\theta = (b-2a)\sin\theta$$

Thus, if the point A were to move along a line through O, the point P would generate an ellipse with axes 2b and 2(b−2a). Attaching A to a Peaucellier cell would insure proper motion. Another simple mechanism is the ordinary connecting rod of a locomotive if the crank is equal in length to the connecting rod. Any point of the rod traces out an ellipse.

If we introduce sliding points in addition to pin joints we can draw all three conics with the arrangement shown in Fig. 3.

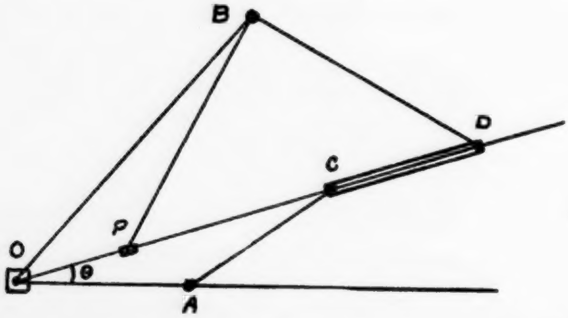


Fig. 3

Let OA = AC = b/2; BP = BD; CD = a. The section CD and the point P are sleeves or slides of some sort. Since OAC is isosceles:

$$OC = b \cos \theta$$

and

$$OD = a + b \cos \theta$$

The three bars OB, BP, BD constitute a Peaucellier cell in which (OP)(OD) = k^2, a constant. If the rod OA is held fixed and OD rotated, then at all times:

$$OP = r = k^2/(a + b \cos \theta)$$

But this is of the form $r = ep/(1 + e \cos \theta)$, the polar equation of a conic. If a set screw be placed at A (to regulate the relative magnitudes of a and b) the eccentricity may be made less than, greater than, or equal to one to draw either the ellipse, hyperbola, or parabola. Without the Peaucellier cell the point D traces the Limacon when $b \neq a$, and the cardioid when $b = a$. (There are certain mechanical difficulties to be met in producing a large section of the curve. The sleeve might be replaced by a bar sliding in a groove of the rod OD. The point P could then occupy a groove in the other side of the same rod and be free to pass the point C.)

With the help of the Peaucellier cell the Conchoid of Nicomedes may be drawn by the arrangement of Fig. 4, P and P′ tracing out the two branches of the curve.

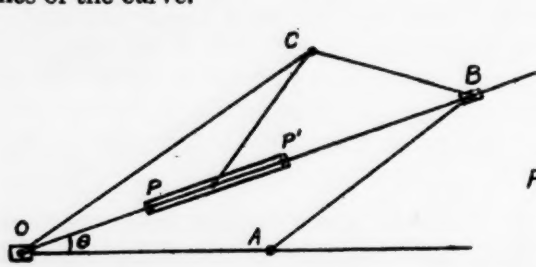


Fig. 4

Let $OA = AB = a/2$; and $PP' = 2b$.

We have: $OB = a \cos \theta$; $OP' = b + k^2/a \cos \theta$ and $OP = -b + k^2/a \cos \theta$

or $$r = \pm b + k^2 \sec \theta / a$$

By selecting different lengths for the rods OA, AB, and the slide PP′ we may obtain the curve with or without the loop.

The curious outfit in Fig. 5 will draw the Strophoid.

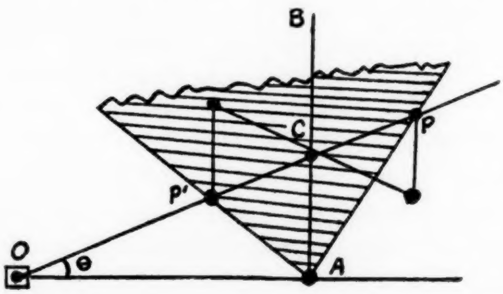


Fig. 5.

The rod AB is fixed at right angles to OA. The points P and P′ are attached to a triangular section (pivoted with its right-angled vertex at A) by grooves in its edges. They are also constrained to the bar OP by slides. A contra-parallelogram is attached to P, P′, and C to insure that C lie always midway between P and P′. (The point C is not attached to the triangle). As the bar OP is rotated about OA the lengths CP, CP′, and CA remain equal to each other. Let $OA = a$. We have:

$$OP' = OC - P'C = OC - CA = a(\sec \theta - \tan \theta)$$

$$OP = OC + CP = OC + CA = a(\sec \theta + \tan \theta).$$

Thus $r = a(\sec \theta \pm \tan \theta)$, the polar equation of the Strophoid.

III

In constructing curves, a great many be formed without the tiresome labor of computing tables of values. The familiar constructions of the text books are useful and the following are given for their variety and entertainment.

1. CD is a variable half-chord parallel to a fixed radius OB of a circle. The locus of the intersection, P, formed by OD and BC is a ***parabola.***

2. AB is a diameter of a circle. Variable chords, CD, are drawn perpendicular to AB. The locus of the intersection of AC and BD is an ***equilateral hyperbola.***

3. Let the radius of a circle be 5a. A point A is taken on a fixed radius at a distance 4a from the center. Other radii are drawn and on them points P are located equidistant from the point A and the circle. The locus of such points P is an ***ellipse.***

4. The polar equation of the Cissoid, $r = 2a \tan \theta \sin \theta$ leads to the following construction. A fixed radius and its perpendicular tangent are drawn to a circle of radius 2a. A line, OC, parallel to the tangent is drawn through the center. A variable radius meets the tangent in a point B. The intercepted length is projected on the line OC and then projected on the variable radius. Points so determined form the *Cissoid.*

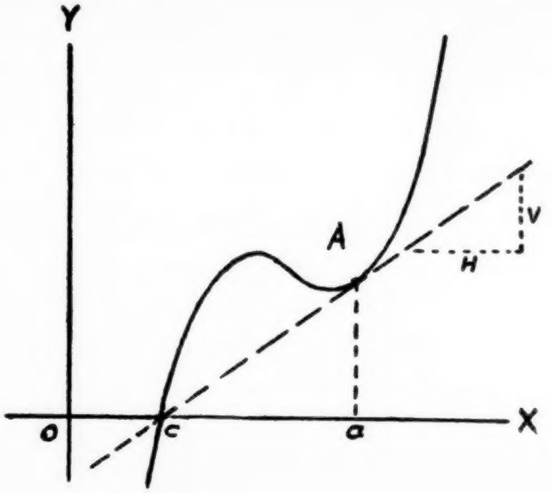
The idea involved in finding Pedal Curves may open a wide field of interest to the student and once established it becomes a source of distinct pleasure. Since tangents to the conics offer no difficulty of construction we present only the pedals that have these as original curves:

Original Curves.	*Pedal Point.*	*Pedal Curve.*
$x^2+y^2=a^2$.	Any Point.	Limacon.
$x^2+y^2=a^2$.	Point on Circumference.	Cardioid.
$x^2+y^2=a^2$.	Center.	The circle itself.
$y^2+4ax=0$.	(2a,0)	Strophoid.
$y^2+4ax=0$.	Vertex.	Cissoid.
$y^2+4ax=0$.	Focus.	Tangent at Vertex.
$b^2x^2 \mp a^2y^2=a^2b^2$.	Focus.	Auxiliary Circle.
$x^2-y^2=a^2$.	Origin.	Lemniscate.

A Graphical Solution for the Complex Roots of a Cubic

By GEORGE A. YANOSIK
New York University

Many times during a discussion of graphing of a polynomial f(x), after showing that the abscissas of the intersections of the graph with the axis of x represent graphically the real roots of the equation f(x) = 0, we are asked by the students whether we can read from the graph the complex roots of the equation. A search of the mathematics literature discloses very little information on this point; consequently, the writer takes this opportunity to offer on this subject a theorem which either is new or certainly not well-known to teachers of algebra.



Confining our attention first to the graph of f(x), where f(x) = 0 is a cubic equation with one real root, say x = c, and a pair of complex roots a ± bi, we know of course that the graph y = f(x) will cross the axis of x once only, at x = c, and will have on it a more or less pronounced bulge, as at A in the figure, which indicates the existence of the pair of complex roots. To obtain from the graph the complex roots a ± bi, we have the following THEOREM:

Given the graph of y = f(x), where f(x) = 0 is a cubic equation with roots c, a ± bi. If a straight line be drawn through (c, o) and tangent

to the curve, the abscissa of the point of tangency is a, and b is the square root of the slope (V/H) of this tangent line.

Proof 1.

Let the cubic be

$$y=(x-c)\ (x^2-2ax+a^2+b^2) \tag{1}$$

A line through (c, 0) is

$$y=m\ (x-c) \tag{2}$$

If this line is tangent to (1), we must have equal roots in

$$x^2-2ax+a^2+b^2-m=0 \tag{3}$$

Setting the discriminant equal to zero, we get

$$m=b^2, \text{ or } b=\sqrt{m}$$

and now (3) gives $x=a$, a for the point of tangency.

Proof 2.

Let $y=x^3+p_1\,x^2+p_2\,x+p_3=0$ have roots c, $a\pm bi$.

Then $y=x^3-(2a+c)x^2+(2ac+a^2+b^2)\,x-c(a^2+b^2)$ (1)

If we allow a to vary, the envelope of the family of cubics is readily found to be

$$y=b^2\ (x-c) \tag{2}$$

which is a line drawn through (c, 0) tangent to the cubics.

Hence, $b=\sqrt{m}$

Solving (1) and (2) for x, we get $x=c, a, a$.

This method applies only to a cubic with one real root and not to higher degree equations. The writer hopes to present shortly some results in connection with the higher degrees.

The Teacher's Department

Edited by
JOSEPH SEIDLIN

AS TO TEACHING PROCEDURES

Due to an inferiority complex, probably caused by reading too much pedagogical literature of the anti-mathematical type, certain teachers of mathematics are anxiously looking for a pedagogical Moses who will present to them, on tables of stone, the ten commandments relating to the methods of teaching of their subject. Hence there is a request for supposedly successful teachers to exhibit what they think is their most effective class room procedure in order that others may profit thereby. Under the circumstances this is a natural request, but its granting may be disastrous. For a procedure which is successful in the hands of one individual may be a decided failure in the hands of another; and a procedure effective with one group of students may not work at all well with another group.

But no one can tell whether a procedure is successful or not unless he knows what it is striving to accomplish. This being the case let us assume that we wish to accomplish the following three things in a recitation:

1) To check up on the student's understanding of information previously imparted.

2) To impart new information to the students.

3) To train students in methods of attacking and analysing problems.

To accomplish the above three things, the writer has found the following procedure very helpful:

1) At the beginning of the recitation check up on how well the students have mastered previous assignments. This is done by having the class work certain typical problems at the board, by questions on important points of previous theory, by answering questions relative to the difficulties of the students, etc.

2) Make a general statement of what the lesson is about with a broad outline of the important steps in the development of the theory involved.

3) Give some class exercises, either oral or written, to familiarize the students with the fundamental steps of the advanced assignment.

This is a period of directed study, during which all, or part, of the students are repeatedly stopped and questioned in an attempt to clear up difficult portions of the theory.

4) Assign a certain number of problems to be worked or a certain amount of theory to be studied for the recitation the next day.

In this procedure, item 3) is probably the most important, as it is here that the students are taught to look at problems rather critically and to determine what methods of attack are likely to be advantageous, either from the standpoint of analysis or of ease in computation. Item 1) is also very important, since it is practically useless to expect assigned tasks to be done regularly unless adequate checks are also regularly made.

The time devoted to each of the items will vary somewhat from day to day, but care must be taken not to devote too much time to item 2), as it is extremely easy for the teacher to talk too much.

This procedure seems to embody some of the good points, and we hope omits the bad ones, of the method which Squeers explained to Nicholas Nickelby as follows:

"Where's the second boy" asked Squeers.

"Please, Sir, he is weeding the garden," replied a small voice.

"To be sure," said Squeers, by no means disconcerted, "so he is. B-o-t bot, t-i-n tin, n-e-y ney, bottiney, noun, substantive, a knowledge of plants. When he knows that bottiney is a knowledge of plants he goes and knows 'em. That's our system Nickelby, what do you think of it?

"It's a very useful one at any rate," answered Nicholas significantly.

J. H. WEAVER,
Ohio State University.

NOTE: This is the first of a series of articles on general or specific procedures in the teaching of elementary college mathematics. The editor invites, nay solicits, comments, criticisms, or similar contributions from our readers. It is hoped that thus we may develop a forum for teachers of college mathematics.—J. S.

Mathematical Notes

Edited by
L. J. ADAMS and I. MAIZLISH

Professor A. L. Hill of Peru State Teachers College, Peru, Nebraska reports the following mathematics section meetings of the Nebraska State Teachers Association, held on October 24 and 25:

1. Lincoln. R. O. Severin, President.
   Remedial Work in High School Mathematics. R. B. Thompson.
   Humanized Mathematics. Miss Helen Dunlap.
2. Omaha. William Sudman, President.
   A New Method of Teaching Geometry. Marian Dodderer.
   Mathematics in Secondary Education Forty Years Ago. Prof. E. M. Stahl.
   The Place of Mathematics in the Changing Curriculum. J. A. Jimerson.
   Mathematics for the Pupil. Miss Eva O'Niell.
3. Norfolk. Miss Jennie Walker, Chairman.
   A Point of View in Teaching. Dr. Clyde M. Hill.
   Motivation in Geometry. Miss Mildred I. Clark.
   Motivation in Junior High School Mathematics. Miss Abbie Bysong.
   Modern Trends in Geometry. Miss Ruth Calendar.
4. Hastings. Eva Phalen, Chairman.
   Address. Dr. A. R. Congdon.
   The Motivation of Plane Geometry. Miss Helen Exley.
5. Holdredge. Allen Lichtenberger, Chairman.
   The Place of Mathematics and Science in Our Changing Curriculum. Dr. Willard W. Patty.
6. Sidney. W. L. Nicholas, President.
   Why Mathematics. Frank Barta.
   Reorganization and Proper Emphasis of Mathematics in Junior High. Round Table.

An electrical machine designed to solve sets of simultaneous linear equations in large numbers of unknowns is proving useful to statisticians, among others. The machine is the invention of R. R. M. Mallock, of England. It is described in the Proceedings of the Royal Society of London, series A, volume 140 (1933) on pages 457-483.

The mathematics department of San Diego State College, of California, is conducting a supervised class in high school algebra, plane and solid geometry, and plane and spherical trigonometry. In this class some eighty students are engaged in reviewing or taking for the first time these elementary courses in order to become better prepared for the mathematical demands of their technical and professional classes. The experiment is under the direction of George R. Livingston, head of the mathematics department.

The Mathematical Association of England published in 1934 a *Report on The Teaching of Algebra in Schools.*

At the annual meeting of the American Mathematical Society, held in St. Louis, Missouri, the following addresses were delivered:

1. *Some Recent Investigations Concerning the Sections of Trigonometric and Related Series.* Professor Szegö.
2. *Tensorial Methods in Dynamics.* Professor Synge.
3. *Linear Differential Equations of Infinite Order.* Professor Carmichael.
4. *Mechanical Analysis.* Professor Vannevar Bush.

The annual convention of the Central Association of Science and Mathematics Teachers heard the following lectures:

1. *Relation of Mathematics to Art, Poetry and Music.* Dr. G. D. Birkhoff, Harvard University.
2. *The Future of Mathematics.* Dr. E. J. Moulton, Dean of the Graduate School, Northwestern University.
3. *Geometrical Constructions Without the Compasses.* Walter H. Carnahan, Shortridge High School, Indianapolis.
4. *Non-Technical Functions of Mathematics.* Dr. Mayme I. Logsdon, University of Chicago.

In addition, there were many lectures on scientific subjects.

A newcomer to the field of mathematical statistics is the journal published by the Indian Statistical Institute, Statistical Laboratory, Presidency College, Calcutta. The first volume was issued in 1934.

Each volume is to consist of four parts, of a total of approximately four hundred quarto pages. The editor is P. C. Mahalanobis.

The American Mathematical Society has published this year, in its Colloquium Series, *Interpolation and Approximation by Rational Functions in the Complex Domain,* by J. L. Walsh.

The Tenth Yearbook of the National Council of Teachers of Mathematics is entitled *The Teaching of Arithmetic.* It contains thirteen chapters, each of which is written by an expert or experts in education, psychology or mathematics.

Professor E. T. Bell, of the California Institute of Technology, addressed the Southern California Science and Mathematics Association at their annual meeting December 17. His subject was *A Guiding Clue Through the History of Mathematics.*

A little known but very valuable mathematics magazine is *Matematica Elemental,* printed under the auspices of La Sociedad Matemática Argentina y de La Sociedad Matemática Española, in Madrid at Duque de Medinacelli, número 4. It is dedicated to undergraduate mathematics, and each monthly issue contains brief notes and comments, short research papers and an unusually lengthy problem department.

L. J. ADAMS.

Problem Department

Edited by

T. A. BICKERSTAFF

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

While it is our aim to publish problems of most interest to the readers, it is believed that regular text-book problems are, as a rule, less interesting than others. Therefore, other problems will be given preference when the space for problems is limited.

Send all communications about problems to T. A Bickerstaff, University, Mississippi.

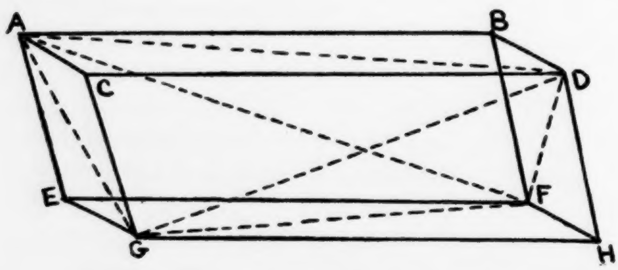
SOLUTIONS

No. 99. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Prove that the sum of the edges of a parallelopiped is smaller than the sum and greater than half the sum of the diagonals of its faces.

Solution by *Ruth Rice*, Norman, Oklahoma.

Consider the parallelopiped ABCDEFGH and the inscribed tetrahedron ADFG.



The sum of the bimedians of the tetrahedron is smaller than one half the sum of the edges of the tetrahedron. (Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 56, MacMillan, 1935.)

But the bimedians of a tetrahedron are equal to the three edges of the circumscribed parallelopiped which pass through a vertex. (ibid., p. 59.)

Hence:
$$AB+AC+AE<\tfrac{1}{2}(AD+AF+AG+DF+DG+FG)$$
$$DC+DB+DH<\tfrac{1}{2}(AD+AF+AG+DF+DG+FG)$$
$$GH+GE+GC<\tfrac{1}{2}(AD+AF+AG+DF+DG+FG)$$
$$FE+FH+FB<\tfrac{1}{2}(AD+AF+AG+DF+DG+FG)$$
$$\text{Sum of edges}<\tfrac{1}{2}(4AD+4AF+4AG+4DF+4DG+4FG)$$

But $AD=EH, AF=CH, AG=BH, DF=EC, DG=BE, FG=BC$

Hence:
$$\text{Sum of edges}<\tfrac{1}{2}(2AD+2EH+2AF+2CH+2AG+2BH+2DF+2EC+2DG+2BE+2FG+2BC)$$
$$\text{Sum of edges}<(AD+EH+AF+CH+AG+BH+DF+EC+DG+BE+FG+BC)$$

Therefore: Sum of edges < Sum of diagonals of faces

The sum of the bimedians of the tetrahedron is greater than one-fourth of the sum of the edges of the tetrahedron. (ibid., p. 56.)

Hence:
$$AB+AC+AE>\tfrac{1}{4}(AD+AF+AG+DF+DG+FG)$$
$$DC+DB+DH>\tfrac{1}{4}(AD+AF+AG+DF+DG+FG)$$
$$GH+GE+GC>\tfrac{1}{4}(AD+AF+AG+DF+DG+FG)$$
$$FE+FH+FB>\tfrac{1}{4}(AD+AF+AG+DF+DG+FG)$$
$$\text{Sum of edges}>\tfrac{1}{4}(4AD+4AF+4AG+4DF+4DG+4FG)$$
$$\text{Sum of edges}>\tfrac{1}{4}(2AD+2EH+2AF+2CH+2AG+2BH+2DF+2EC+2DG+2BE+2FG+2BC)$$
$$\text{Sum of edges}>\tfrac{1}{2}(AD+EH+AF+CH+AG+BH+DF+EC+DG+BE+FG+BC)$$

Therefore: Sum of edges > $\frac{1}{2}$ sum of diagonals of faces.

Also solved by *A. C. Briggs*, and the *Proposer*.

No. 100. Proposed by W. V. Parker, Georgia Tech.

If P_1, P_2, P_3, P_4 are the incenter and the three excenters of a triangle, the system of conics on them is a system of equilateral hyperbolas with centers on the circumscribing circle of the triangle.

Solution by *Karleton W. Crain*, Purdue University.

Let P_2, P_3, and P_4 be the excenters opposite the vertices A, B, and C respectively of the triangle ABC. Now P_1 is the intersection of the altitudes of triangle $P_2P_3P_4$ as well as the incenter of triangle ABC. If we choose P_1P_2 as the x-axis and P_3P_4 as the y-axis, we may let the rectangular coordinates of the points be $P_2(x_2,0)$, $P_3(0,\ y_3)$, $P_4(0,\ y_4)$ and then P_1 would have the coordinates $P_1(-y_3y_4/x_2,0)$.

The equation of P_2P_3 is $xy_3+yx_2-x_2y_3=0$, and the equation of P_4P_1 is $xx_2-yy_3+y_3y_4=0$. Then $xy=0$, and $(xx_2-yy_3+y_3y_4)(xy_3+yx_2-x_2x_3)=0$ are two conics through the four points P_1,P_2,P_3,P_4.

Now $$\lambda xy+(xx_2-yy_3+y_3y_4)(xy_3+yx_2-x_2y_3)=0 \tag{1}$$

represents all the conics of the system.

If equation (1) be rearranged, we have

$$x^2(x_2y_3)+y^2(-x_2y_3)+xy(\lambda+x_2^2-y_3^2)+x(y_3^2y_4-x_2^2y_3)$$
$$+y(x_2y_3^2+x_2y_3y_4)-x_2y_3^2y_4=0.$$

Since the sum of the coefficients of x^2 and y^2 is zero, all the conics of this system are equilateral hyperbolas.

Taking the partial derivatives of equation (1), first with respect to x and then with respect to y we get the following equations which give the coordinates of the center,

$$\lambda y+(xx_2-yy_3+y_3y_4)y_3+(xy_3+yx_2-x_2y_3)x_2=0$$

$$\lambda x+(xx_2-yy_3+y_3y_4)x_2-(xy_3y+x_2-x_2y_3)y_3=0.$$

Multiplying these by x and y respectively and subtracting we have, for all values of λ,

$$2x_2y_3x^2+2x_2y_3y^2+x(y_3^2y_4-x_2^2y_3)+y(-x_2y_3^2-x_2y_3y_4)=0$$

which is the locus of centers of the equilateral hyperbolas. This circle is the Nine Point Circle of triangle $P_2P_3P_4$ since it passes through the mid-points of the sides. (This may be verified by substituting the co-ordinates of these points into the above equation.) Since the N. P. C. of the triangle also passes through the feet of the altitudes, this circle passes through the points A, B, and C and is therefore, the circumference of triangle ABC.

Also solved by *A. C. Briggs*.

No. 101. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Construct a sphere passing through a given point so that the distances of its center from three given points shall be equal, respecttively, to a pth, qth, and rth part of the radius.

Note. The corresponding problem in the plane was discussed in the *Educational Times, Reprints,* Vol. XXXVI (1881), p. 49, Q. 5592.

Solution by the *Proposer.*

Let A be the point on the sphere and P the first of the other three given points. If 0 and a are the center and the radius of the sphere, we have, by assumption,

$$OA : OP = a : pa = 1 : p$$

Thus the point 0 belongs to a sphere of Apollonius*.

Similarly for the other two given points Q, R. Consequently the point 0 lies on three known spheres and therefore is determined. The problem may have two solutions.

Also solved by *A. C. Briggs.*

No. 102. Proposed by Norman Anning, University of Michigan.

Prove that any one of the points $(lm,0)$, $(mn,0)$, $(0,nl)$, $(0,-m^2)$ is the ortho-center of the triangle which has the other points for vertices.

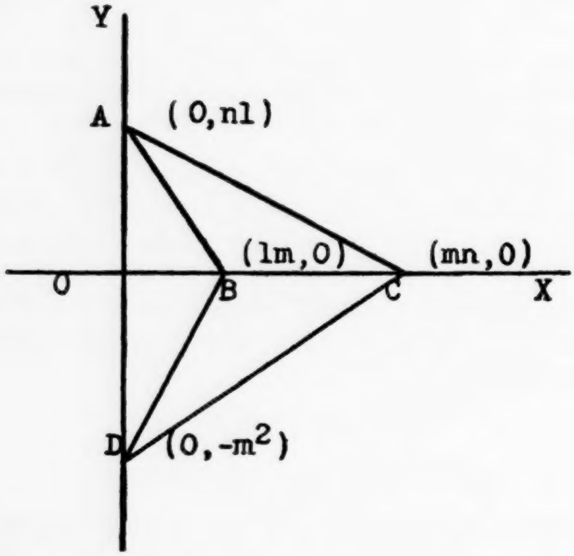
I. Solution by *R. C. Yates,* University of Maryland.

It is well known that the vertices of any triangle together with the orthocenter form an orthocentric group. The triangle $A:(0,nl)$, $B:(0,-m^2)$, $C:(mn,0)$ has for one altitude the X-axis and for another the line $nx+my=lmn$. These cross at $(lm,0)$. Q. E. D.

Furthermore, A, B, C, determine the family of rectangular hyperbolas $x^2+Bxy-y^2-m(n+l)x+(nl-m^2)y+lnm^2=0$. Every curve of this family contains the point $(lm,0)$. Therefore, the four points form an orthocentric group. (See Problem 93, Vol. 10, No. 1.)

*Nathan Altshiller-Court, Modern Pure Solid Geometry, p. 5. Macmillan, 1935.

II. Solution by *A. C. Briggs.*



The equation to line,

$$\begin{array}{ll} AB, & y=-nx/m+nl; \\ AC, & y=-lx/m+nl; \\ BD, & y=mx/l-m^2; \\ CD, & y=mx/n-m^2; \\ BC, & y=0;\ AD, -x=0. \end{array}$$

Note that AD and BC are perpendicular; likewise are AB and CD, and AC and BD.

In the triangle ABD, AC is perpendicular to BD and CD to AB, and therefore their intersection, C, is the ortho-centre of ABD. For the same reason, B is the ortho-centre of ACD.

Since AC is perpendicular to BD and AD to BC, their intersection, A, is the ortho-centre of BCD. And since BD is perpendicular to AC and AD to BC, their intersection, D, is the ortho-centre of ABC.

Q. E. D.

No. 103. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The sum of the powers of a point with respect to four given spheres is equal to two thirds of the sum of the powers of the same points

with respect to the six radical spheres* of the given spheres taken two at a time.

Note. The corresponding proposition relative to two circles was discussed in the *Educational Times, Reprints,* Vol. II (1865), p. 67, Q. 1540.

Solution by the *Proposer.*

Let a, b, (ab) be the powers of the given point P with respect to the two given spheres (A), (B) and with respect to their radical sphere (AB). If R is the center of (AB) and U the foot of the perpendicular from P upon the common radical plane of the three spheres, we have, both in magnitude and in sign,†

$$a-(ab)=2AR\cdot UP,\quad b-(ab)=2BR\cdot UP$$

hence $$a+b-2(ab)=2UP(AR+BR)=2UP\cdot 0=0$$

or $$a+b=2(ab).$$

In a similar way we may obtain five other analogous relations. Adding these six relations we obtain the announced result.

PROBLEMS FOR SOLUTION

No. 110. Proposed by Walter B. Clarke, San Jose, Cal.

The angles of a triangle are $A<B<C$ and C' is the midpoint of AB. A cevian from B to AC at D has its segments outside the incircle equal. Show that bisector of the angle at C' of the medial triangle of triangle ABC is perpendicular to BD.

No. 111. Proposed by Walter B. Clarke.

O and O' are centers of two unequal circles intersecting each other at S and T. Line OO' cuts the circles at D, F, G, and E in that sequence. Show that

$$\sqrt{DF\cdot FG\cdot GE\cdot DE}=OO'\cdot ST$$

No. 112. Proposed by Walter B. Clarke.

Show that the distance of the orthocenter from the longest side of a triangle equals the sum of the distances of incenter, verbicenter, and Nagal Point from that side.

*Nathan Altshiller-Court, Modern Pure Solid Geometry, p. 173. Macmillan, 1935.
†Ibid., p. 183.

(The verbicenter is the concurrent point of lines from each vertex to the point half way around the perimeter. Nagel Point is the concurrent point of perpendiculars from each excenter to the nearest side.)

No. 113. Proposed by W. Van Parker, Georgia Tech.

If k is any given integer >0, show that the equation

$$(x+y)^2+(x-y)=2K$$

has a unique solution in integers such that $x \geqq 0$, $y > 0$

Book Reviews

Edited by

P. K. SMITH

Modern Pure Solid Geometry. By Nathan Altshiller-Court. New York. The Macmillan Company. Bound—cloth. Crown—8vo. 1935. x plus 311 pages. $3.90.

Mathematicians and their friends will be delighted with *Modern Pure Solid Geometry,* just off the press—very attractively bound in beautiful cobalt blue, of nice size, written in a clear style and of high merit.

The author, in referring to the possible faults of his book, says, "The author offers all due apologies for their barbarity," and again, "I am painfully aware of the book's inevitable shortcomings." But his book calls for no apologies: it is an excellent treatise; it will fill a long-felt need, and certainly it is the only thing of its kind accessible to students of modern geometry.

The volume is a logical sequence of Dr. Altshiller-Court's very successful work *College Geometry,* 1925. *College Geometry,* also, was a new venture, but *Modern Pure Solid Geometry,* of course, is a much further venture and a more exacting one. Its material, too, has had to be gathered from isolated and more unrelated sources, both new and old, an even larger proportion of such material having never appeared before in book form, scarcely any of it in the English. Obviously, it has been a real task to weld this material into an acceptable work.

While presumably the only prerequisite to a study of the book is "a knowledge of the elements of solid geometry and some acquaintance with modern plane geometry", yet the author specifically implies that a good deal of real mathematical maturity may be needed "in many places."

Conic sections are not treated—only the point, line, plane, circle, and sphere are included; their treatment is "exclusively synthetic". The harmonic ratio is used freely; also, the imaginary sphere is used. But notions like the anharmonic ratio, involution, complete quadrilateral and elements at infinity do not appear.

The main body of the material is broken into nine chapters: Preliminary, The Trihedral Angle, The Skew Quadrilateral, The Tetrahedron, Transversals, The Oblique Cone with Circular Base, Inversion, and Recent Geometry of the Tetrahedron. Detailed subdivisions of these chapters are contained in a circular which may be secured from the publishers merely by dropping them a card requesting such information.

The book is well supplied with exercises. Many are easily solved, but a few will challenge the best efforts of the students.

The text does not include such a great number of figures. The reviewer wonders whether a new field, with its shortage of experienced teachers, will not need a more generous supply of drawings. However, the figures are well drawn and make a strong appeal to the student. Perhaps a slight improvement in certain ones may be secured by dotting all lines not visible to the eye of the observer; for example, lines AC, SH in figure 12; lines AC, AA′, CC′ in figure 15, and similar cases in many of the figures. If such be *given* lines, they may be dotted, but of a different weight from the dotted lines auxiliary to the proof.

The type used and the arrangement of matter on the page also deserve special mention—their effect is very pleasing. A second kind of slight objection may be noted on page 21 where a theorem is permitted to fall at the bottom of the page, whereas its accompanying proof with its figure falls on page 22; similarly in a few other cases. Such separation of material is retarding as the eye passes from the theorem to the proof, and back again, in rapid glances.

Such slight objections may be easily removed in subsequent reprints—they are offered only as "suggestions for future use."

The book contains an abundance of material for a three hour course for two semesters, but it may be considerably abridged. The author has been mindful of this frequent need, and hence has arranged his material so that certain omissions may be accepted without hindrance to the general course of study. These omissions are suggested in the preface.

A splendid six-page index is added. Included also is a good three-page bibliography, a thing of joy to the persons reaching out for a further study of the subject.

Necessarily the book bears the distinctive stamp of the author. He himself has refined its material through repeatedly using it in his own class room. Moreover, as mentioned in the beginning, since there were no paths already laid out along which he could stroll with ease, he had little to guide him save his own experience. But he has done a most difficult job and has executed it in a fashion that will meet with the highest endorsement of mathematicians everywhere.

Personally the reviewer has thoroughly enjoyed teaching Dr. Altshiller-Court's *College Geometry* since its appearance in mimeograph form, and now feels a keen thrill in having before him this new treatise on *Solid Geometry*. What a delight it will be to go through the book, step by step, with a class of students ready for some real adventure in this realm of modern mathematics!

IRBY C. NICHOLS.